Dark Matter Answers
Sutton Trust Summer School, July 2015

Part I: the evidence for dark matter

(a) The gravitational force between the two masses is,

$$F = \frac{GMm}{r^2} \tag{1}$$

This the force that is sourcing the centripetal motion so another way of writing F is the formula for centripetal force,

$$F = \frac{mv^2}{r} \tag{2}$$

So equating these two forces we can rearrange for v, notice how the smaller mass m cancels,

$$\frac{GMm}{r^2} = \frac{mv^2}{r} \tag{3}$$

$$\frac{GM}{r} = v^2 \tag{4}$$

$$\implies v = \sqrt{\frac{GM}{r}} \tag{5}$$

(b) Rearranging the formula for v we find,

$$M = \frac{v^2 r}{G} \tag{6}$$

and plugging in $v = 270,000$ m s^{-1} (careful with units!) and $r = 1.8 \times 10^{21}$ we get $M_{\text{tot}} = 1.97 \times 10^{42}$ kg.

(c) If the sun is an average star then an estimate for the total mass of stars is simply 10^{11} stars multiplied by the mass of the Sun, giving us $M_{\text{stars}} = 2 \times 10^{41}$ kg.

(d) The fraction of the Milky Way that is stars is then $f_{\text{stars}} = M_{\text{stars}}/M_{\text{tot}}$ = 0.1. So if 10% of the Milky Way is stars this implies that around 90% must be dark matter.

There were a few simplifications that we used in this question that we could address if we wanted a more accurate estimate. The first is the use of a simple circular motion model. Not only are the orbits of stars not circular but the formula we used assumes that the star is in orbit around a single mass M, whereas in reality the mass is distributed over the Galaxy. A more refined estimate would use the density profile of the Milky Way which is the density as a function of radius $\rho(r)$.

Another simplification is that we estimated the total mass of stars only using the mass of the Sun. It turns out that this isn't too bad of an approximation as the Sun is a pretty average star if we did a proper calculation we would need to know how many stars were more massive than the sun, and how many were lighter to properly count them, this would need something called an initial mass function.

The last major approximation is in calculating the fraction of dark matter, we assumed that all of the mass that isn't stars is dark matter which is not true. There are other things in the Milky Way such as gas, dust, planets, aliens, etc. not to mention the rather large black hole at the center, all of which should be taken into account when doing a full calculation.

Believe it or not, even with all the approximations that are made in this question, the result is actually pretty good; around 90% of the Milky Way's mass is indeed dark matter. But I should stress that making all of these adjustments to a simple calculation is not easy! Calculating how much dark matter is in the Milky Way is a very active area of research and we definitely don't have the final answer yet. This is how a lot of physics is done though, you start with a simple model or approximation and slowly refine your estimates and make more accurate observations until you arrive at the answer that makes everything fit together.

Part II: dark matter interactions

(a) Number density is defined as the number of particles per unit volume. To get the number of dark matter particles per unit volume we must divide the density of dark matter by the mass of a single dark matter particle which I've told you is $50m_p$, so $n = \rho_{\text{dm}}/50m_p$ works out to be 6300 m^{-3}.

(b) An average human body has a mass of about 70 kg. The mass of a single water molecule is the mass of the 2 protons in the two hygrogen nuclei plus the mass of the 8 neutrons and 8 protons in the oxygen nucleus (the mass of the electron is tiny so we'll ignore them). The total number of oxygen nuclei, which is the same as the number of water molecules is then $70\,\text{kg}/18m_p = 2.3 \times 10^{27}$.

(c) All that remains is to plug the numbers into the formula for R.

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\begin{aligned}
R &= NnA^2\sigma \langle v \rangle & (7)\\
&= (2.3 \times 10^{27}) \times (6300) \times (16^2) \times (10^{-46}) \times (220,000) & (8)\\
&= 8.16 \times 10^{-8}\,\text{s}^{-1} & (9)
\end{aligned}
$$

(d) To calculate the number of collisions in a year we multiply this rate by the number of seconds in a year $60 \times 60 \times 24 \times 365$ which gives us a total number of events per year of 2.57. So on an average year around 2 or 3 dark matter particles will collide with your body. This is quite remarkable given that the answer to part (a) told us that there were 6300 dark matter particles in a cubic metre!